

Monopole Chern-Simons Term: Charge-Monopole System as a Particle with Spin

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Abstract

The topological nature of Chern-Simons term describing the interaction of a charge with magnetic monopole is manifested in two ways: it changes the plane dynamical geometry of a free particle for the cone dynamical geometry without distorting the free (geodesic) character of the motion, and in the limit of zero charge's mass it describes a spin system. This observation allows us to interpret the charge-monopole system alternatively as a free particle of fixed spin with translational and spin degrees of freedom interacting via the helicity constraint, or as a symmetric spinning top with dynamical moment of inertia and “isospin” $U(1)$ gauge symmetry, or as a system with higher derivatives. The last interpretation is used to get the twistor formulation of the system. We show that the reparametrization and scale invariant monopole Chern-Simons term supplied with the kinetic term of the same invariance gives rise to the alternative description for the spin, which is related to the charge-monopole system in a spherical geometry. The relationship between the charge-monopole system and $(2+1)$ -dimensional anyon is discussed in the light of the obtained results.

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1 Introduction

The Dirac charge-monopole system [1] was one of the first models with which the importance of topology for physics was realized. The model was investigated in various aspects classically and quantum mechanically [2]-[13], and the Dirac's quantization of the charge-monopole constant was understood naturally in terms of the underlying topological fibre bundle structure [6],[10]-[12]. In addition, an interesting analysis of the Dirac quantization condition relies on 3-cocycles [13]. Another very seminal interplay of physics and topology is related to the Chern-Simons (CS) “effects” [14]-[29], whose one of the basic physical aspects established by Deser, Jackiw and Templeton [14] consists in quantization of coupling in field theory analogous to Dirac quantization in quantum mechanics.

The spin particle nature of the charge-monopole system was observed by Jackiw, Rebbi, Hasenfrantz and 't Hooft in the context of the “spin from isospin” field theoretical mechanism [2, 3], but the first results on the nature of this system anticipating its interpretation as a particle with spin were obtained by Poincaré (see ref. [7]). In this paper we exploit the CS nature of the charge-monopole coupling term to interpret the charge-monopole system as a particle with spin from the opposite side, not appealing to its field theoretical origin but treating it classically and quantum mechanically as a system with finite number of degrees of freedom. More specifically, we observe that the charge-monopole coupling term having a nature of (0+1)-dimensional CS term manifests its topological nature in two ways. First, the CS term changes the plane dynamical geometry of a free particle for the cone dynamical geometry of a charged particle without distorting the free (geodesic) character of the motion. Second, in the limit of zero charge's mass the CS term describes a spin system. This observation allows us to interpret the charge-monopole system alternatively:

- (i) as a free particle of fixed spin with translational and spin degrees of freedom interacting via the helicity constraint; in such interpretation, noncommuting gauge-covariant momenta (velocities) are the analogs of the Foldy-Wouthuysen coordinates of the Dirac particle;
- (ii) as a symmetric spinning top with dynamical moment of inertia and “isospin” $U(1)$ gauge symmetry;
- (iii) as a system with higher derivatives; this interpretation is the charge-monopole analog of another known classical equivalence [30] between the relativistic scalar massive particle in a background of the constant homogeneous electromagnetic field and higher derivative model of relativistic particle with torsion [31] underlying (2+1)-dimensional anyons [32]-[36];
- (iv) as an analog of the massless particle with nonzero spin, that naturally leads to the twistor formulation for the charge-monopole system.

We also show that the reparametrization and scale invariant CS term supplied with the kinetic term of the same invariance gives rise to the alternative description for the spin. The partially gauge fixed version of such spin model corresponds to the charge-monopole system in a spherical geometry. The obtained results are used to discuss the relationship between the charge-monopole system and (2+1)-dimensional anyon.

The paper is organized as follows. Section 2 is devoted to discussion of the classical theory of the charge-monopole system in the context of its similarity to the 3D free particle. Here we observe the geodesic character of the charge's motion and demonstrate that being reduced to the fixed level of the angular momentum integral, it is described by the Lagrangian corresponding to the 2D non-relativistic particle in the planar gravitational field of a point-

like source [37] carrying simultaneously a nontrivial magnetic flux. This, in particular, explains why the charge-monopole system and 2D charged particle moving in the field of the point magnetic vortex have the same “hidden” or “dynamical” $SO(2,1)$ symmetry revealed by Jackiw [8, 38]. We obtain the general solution to the canonical equations of motion, compare the charge-monopole system with the 3D free particle from the point of view of integrals of motion and their Lie-Poisson algebras, and, finally, interpret the charge-monopole system as a reduced $E(3)$ system. In section 3 we discuss the classical and quantum theory of the spin represented in the form of the $(0+1)$ -dimensional topological theory given by the CS charge-monopole action corresponding to the limit of zero charge’s mass. In section 4 we construct the description of the charge-monopole system as a particle with spin. We start with the free particle system with spin, whose value is fixed by the charge-monopole constant. Then we switch on interaction between translational and spin degrees of freedom by introducing the helicity constraint which freezes spin degrees of freedom and provides finally the physical equivalence of the extended model to the initial charge-monopole system. In section 5 the system is interpreted as a spinning symmetric top. In this picture the initial $U(1)$ electromagnetic gauge symmetry is changed for the $U(1)$ gauge symmetry generated by the “isospin”-fixing constraint, which makes the rotations about the top’s symmetry axis to be unobservable. The spinning top picture is used in section 6 to get the higher derivative form for the CS term, which, in turn, is employed in section 7 for constructing the twistor formulation for the charge-monopole system proceeding from its analogy to the 4D massless particle with spin. In section 8 we discuss the charge-monopole system in a spherical geometry and find the alternative formulation for the spin. Here we also observe that the $SO(2,1)$ symmetry of the charge-monopole system [8, 38] can be treated formally as a relic of the reparametrization invariance surviving the Lagrangian gauge fixing procedure applied to the Euclidean relativistic version of the model. Section 9 contains discussion of the relationship between the charge-monopole system and $(2+1)$ -dimensional anyon, and in the last section we present some concluding remarks.

2 Charge-monopole dynamics and 3D free particle

2.1 Lagrangian formalism

A non-relativistic particle of unit mass and electric charge e in the field of magnetic monopole of charge g is described by the Lagrangian

$$L = \frac{1}{2}\dot{\mathbf{r}}^2 + e\mathbf{A}\dot{\mathbf{r}}, \quad (2.1)$$

with a $U(1)$ gauge potential $\mathbf{A}(\mathbf{r})$ defined by the relations

$$\partial_i A_j - \partial_j A_i = F_{ij} = \epsilon_{ijk} B_k, \quad B_i = g \frac{r_i}{r^3}, \quad r = \sqrt{\mathbf{r}^2}. \quad (2.2)$$

The case of arbitrary mass m can be obtained from (2.2) via the transformation of time and charge: $t \rightarrow m^{-1/2}t$, $e \rightarrow m^{1/2}e$. In definition (2.2) it is assumed that the point $r = 0$ is excluded, i.e. the configuration space of the system, $\mathcal{M} = \mathbb{R}^3 - \{0\}$, is diffeomorphic to $(0, \infty) \times S^2$ and inherits a nontrivial topology of a two-sphere S^2 . It is well known that the

electromagnetic potential (2.2) gives a connection of the monopole U(1) fibre bundle being a nontrivial Hopf bundle over S^2 , and the problems with Dirac strings of singularity can be escaped by covering the configuration space \mathcal{M} with two charts [39, 27]. These topological complications, however, do not play any role for us under treating the classical dynamics of the system, but will reveal themselves at the quantum level.

Equations of motion following from Lagrangian (2.1) result in the Lorentz force law,

$$\ddot{\mathbf{r}} = -\frac{\nu}{r^3} \cdot \mathbf{r} \times \dot{\mathbf{r}}, \quad \nu = eg, \quad (2.3)$$

which implies that instead of the orbital angular momentum vector $\mathbf{L} = \mathbf{r} \times \dot{\mathbf{r}}$, the vector

$$\mathbf{J} = \mathbf{L} - \nu \mathbf{n}, \quad \mathbf{n} = \mathbf{r} \cdot r^{-1}, \quad (2.4)$$

is the conserved angular momentum of the system. The nontrivial term $-\nu \mathbf{n}$ can be understood as the electromagnetic angular momentum produced by both the electric charge and magnetic monopole (see ref. [7]). Note that $J = \sqrt{\mathbf{J}^2}$ is restricted from below by the modulus of the charge-monopole coupling constant ν , $J \geq |\nu|$. Due to the relation $\mathbf{J}\mathbf{n} = -\nu$, the trajectory of the particle lies on the cone. The cone's axis is given by the vector \mathbf{J} and its half-angle is

$$\cos \gamma = -\nu J^{-1}. \quad (2.5)$$

Since the force $\mathbf{f} = -\nu r^{-3} \mathbf{r} \times \dot{\mathbf{r}}$ is orthogonal to \mathbf{r} and to the velocity $\dot{\mathbf{r}}$, it is perpendicular to the cone. Therefore, the particle performs a *free motion on the cone*. This can also be observed directly as follows. Taking into account the conservation of the angular momentum \mathbf{J} and that in spherical coordinates the charge-monopole coupling term takes a form $L_{int} = \nu \cos \vartheta \dot{\varphi}$ (see Eq. (3.9)), we can choose the system of coordinates with axis $\vartheta = 0$ directed along the vector \mathbf{J} . Then in accordance with Eq. (2.5), $\cos \vartheta = -\nu J^{-1}$, $\dot{\vartheta} = 0$, and Lagrangian (2.1) is reduced to $L = \frac{1}{2}(\dot{r}^2 + (1 - \nu^2 J^{-2})r^2 \dot{\varphi}^2) - \nu^2 J^{-1} \dot{\varphi}$. After transformation

$$t \rightarrow t' = \alpha t, \quad \alpha = \sqrt{1 - \nu^2 J^{-2}}, \quad (2.6)$$

we obtain finally the following form for the charge-monopole Lagrangian:

$$L = \frac{1}{2}(\alpha^{-2} \dot{r}^2 + r^2 \dot{\varphi}^2) - \alpha \nu^2 J^{-1} \dot{\varphi}. \quad (2.7)$$

The last term is the total time derivative of the topologically nontrivial angular variable. It is the reduced form of the charge-monopole interaction term having a nature of $(0+1)$ -dimensional CS term [22]-[26]. As we shall see, quantum mechanically this will give rise to the Dirac quantization condition for ν , but classically the total derivative can be omitted without changing the equations of motion. The topologically nontrivial term corresponds exactly to the 2D term describing the interaction of the charge e with a (singular) point vortex carrying the magnetic flux [38] $\Phi = -2\pi\alpha\nu^2 J^{-1}e^{-1}$. Without the last term, Eq. (2.7) is a Lagrangian of a *free* particle of unit mass on the cone given by the relations $x = r \cos \varphi$, $y = r \sin \varphi$, $z = r\sqrt{\alpha^{-2} - 1}$, $r > 0$, $0 \leq \varphi \leq 2\pi$, with $0 < \alpha < 1$, that confirms our statement on a free (geodesic) motion of the charge over the cone (2.5). Together with the pointed out nature of the total derivative term, this, as we shall see, explains why the charge-monopole

system and 2D charged particle moving in the field of the point magnetic vortex have the same dynamical $SO(2,1)$ symmetry [8, 38].

Since the conical metric $ds^2 = \alpha^{-2}(dr)^2 + r^2(d\varphi)^2$ corresponds to the metric produced by the point mass [37, 40], we can say that the classical motion of the charge in the field of magnetic monopole (reduced to the fixed value of the integral \mathbf{J}) is equivalent to the classical motion of a particle in a planar gravitational field of a point massive source carrying simultaneously magnetic flux $\Phi = -2\pi\alpha\nu^2 J^{-1}e^{-1}$.

From the form of transformation (2.6) and Lagrangian (2.7) it is clear that the case $J = |\nu|$ is singular and should be treated as a limit case, i.e. we have to assume that $J > |\nu|$. We shall discuss this peculiarity of the charge-monopole system in different aspects in what follows.

In the case of a 3D free particle ($\nu = 0$, $\mathcal{M} = \mathbb{R}^3$), the motion is characterized by the coordinate \mathbf{r} and by the conserved linear momentum \mathbf{p} . Alternatively, with the appropriate choice of the origin of the system of coordinates, the particle's motion can be characterized by the unit vector \mathbf{n} and by the conserved orbital angular momentum \mathbf{L} supplemented with the canonically conjugate scalars r and $p_r = \mathbf{p} \cdot \mathbf{r}^{-1}$. Since $\mathbf{L}\mathbf{n} = 0$, for a given \mathbf{L} the particle's trajectory is in the plane orthogonal to the orbital angular momentum. So, we conclude that classically the topological nature of $(0+1)$ -dimensional charge-monopole CS term is manifested in changing the global structure of the dynamics without distorting its local free (geodesic) character: the “plane dynamical geometry” of the free particle ($e = 0$) is changed for the free “cone dynamical geometry” of the charged particle. We shall discuss the relation between the two systems in the context of (dynamical) integrals of motion in subsection 2.3.

2.2 Canonical formalism: solutions to the equations of motion

To solve the equations of motion in general form and analyze in more detail the system's dynamics, we turn to the canonical formalism. The Hamiltonian corresponding to Lagrangian (2.2) is $H = \frac{1}{2}\mathbf{P}^2$, where $\mathbf{P} = \mathbf{p} - e\mathbf{A}$ is a classical analog of the gauge-covariant derivative defined via the momentum \mathbf{p} canonically conjugate to \mathbf{r} . This gives rise to the Poisson brackets

$$\{P_i, P_j\} = \frac{\nu}{r^3}\epsilon_{ijk}r_k, \quad \{r_i, P_j\} = \delta_{ij}, \quad \{r_i, r_j\} = 0, \quad (2.8)$$

and to the equations of motion

$$\dot{\mathbf{r}} = \mathbf{P}, \quad \dot{\mathbf{P}} = -\frac{\nu}{r^3}\mathbf{L}, \quad (2.9)$$

with $\mathbf{L} = \mathbf{r} \times \mathbf{P}$. From Eq. (2.9) we find that the vectors \mathbf{n} and \mathbf{L} evolve according to the equations $\dot{\mathbf{L}} = \nu r^{-2} \cdot \mathbf{L} \times \mathbf{n}$, $\dot{\mathbf{n}} = r^{-2} \cdot \mathbf{L} \times \mathbf{n}$, and, as a consequence, precess about the integral $\mathbf{J} = \mathbf{L} - \nu\mathbf{n}$ with the same frequency: $\dot{\mathbf{L}} = r^{-2} \cdot \mathbf{J} \times \mathbf{L}$, $\dot{\mathbf{n}} = r^{-2} \cdot \mathbf{J} \times \mathbf{n}$. The vectors \mathbf{J} and \mathbf{n} obeying the relations $\mathbf{n}^2 = 1$ and $\mathbf{J}\mathbf{n} = -\nu$, together with the two scalars \mathbf{r}^2 and $\mathbf{P}\mathbf{r}$ form the complete set of observables in terms of which \mathbf{r} and \mathbf{P} can be completely “restored”. The equations of motion for the scalar observables have a simple form $(\mathbf{r}^2) = 2\mathbf{P}\mathbf{r}$, $(\mathbf{P}\mathbf{r}) = \mathbf{P}^2 = 2H$, being exactly the same as in the case of a free particle. Their integration gives

$$(\mathbf{P}\mathbf{r})(t) = (\mathbf{P}\mathbf{r})(t_0) + \mathbf{P}^2 \cdot (t - t_0), \quad \mathbf{r}^2(t) = \mathbf{P}^2 \cdot (t - t_0)^2 + 2(\mathbf{P}\mathbf{r})(t_0) \cdot (t - t_0) + \mathbf{r}^2(t_0). \quad (2.10)$$

The minimal charge-monopole distance corresponds to the moment of time for which $\mathbf{Pr} = 0$, and is given by the relation $\mathbf{r}_{\min}^2 = \mathbf{L}^2 \cdot \mathbf{P}^{-2}$. To complete the integration of equations of motion, we solve the equation $\dot{\mathbf{n}} = r^{-2} \cdot \mathbf{J} \times \mathbf{n}$ and get

$$\mathbf{n}(t) = -\nu J^{-1} \mathbf{j} + \mathbf{n}_{\perp}(t), \quad \mathbf{n}_{\perp}(t) = (\mathbf{n}(t_0) + \mathbf{j} \nu J^{-1}) \cos \tau(t) + \mathbf{j} \times \mathbf{n}(t_0) \sin \tau(t), \quad (2.11)$$

$$\tau = J L^{-1} \cdot \tan^{-1}(\mathbf{Pr} \cdot L^{-1}), \quad L = \sqrt{\mathbf{J}^2 - \nu^2}, \quad (2.12)$$

where $\mathbf{j} = \mathbf{J} \cdot J^{-1}$, and evolution of \mathbf{Pr} is given by Eq. (2.10). Zero value of the angular function $\tau(t)$ corresponds to the point of perihelion ($r = r_{\min}$) with respect to which the trajectory is symmetric. The vector \mathbf{n}_{\perp} is the projection of \mathbf{n} to the plane orthogonal to the angular momentum \mathbf{J} , and Eqs. (2.11), (2.12) give the classical scattering angle of the particle's motion projected into the plane perpendicular to \mathbf{J} as a function of it:

$$\varphi_{\text{scat}}^{\perp} = \tau(+\infty) - \tau(-\infty) = \pi J L^{-1}. \quad (2.13)$$

Eq. (2.11) together with Eq. (2.10) and relation $\mathbf{P} = \dot{\mathbf{r}}$ give a complete solution to the equations of motion (2.9).

In correspondence with Eqs. (2.5), (2.13), in the limit $J \rightarrow \infty$ the cone over which the particle moves is close to the plane: $\gamma \rightarrow \pi/2$ and $\varphi_{\text{scat}}^{\perp} \rightarrow \pi$. In another limit, $J \rightarrow |\nu|$, the cone is degenerated into a half-line, $\gamma \rightarrow 0(\pi)$ for $\nu < 0(> 0)$, whereas the number of full rotations $\mathcal{N} = [\varphi_{\text{scat}}^{\perp}/2\pi]$ is infinite, where $[.]$ is the integer part. Therefore, the case $J = |\nu|$ ($\mathbf{L} = 0$) with $\mathbf{P} \neq 0$, which corresponds to the motion of the charge with constant velocity along a straight half-line defined by $\mathbf{n}(t_0)$ in the direction to or from the monopole is not a proper limit: such a trajectory corresponds to the half of the trajectory with $J \rightarrow \infty$ ($L \rightarrow \infty$) but not to the limit $J \rightarrow |\nu|$. This supports our conclusion of the previous subsection on a necessity to treat the values of the angular momentum to be confined to the domain $|\nu| < J < \infty$.

2.3 Integrals of motion and their algebra

Let us compare the charge-monopole system with a 3D free particle from the point of view of integrals of motion and corresponding Lie-Poisson algebras formed by them.

For the 3D free particle ($\nu = 0$), the first integrals of motion are linear, \mathbf{p} , and angular, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, momenta forming the set of 5 ($\mathbf{pL} = 0$) algebraically independent conserved quantities not depending explicitly on time. With respect to the Poisson brackets, they form the algebra of Euclidean group $E(3)$. Algebraically, the Hamiltonian is a square of the vector \mathbf{p} , $H = \frac{1}{2}\mathbf{p}^2$, but it generates the symmetry of time translations being independent from the symmetries of space translations and rotations generated by \mathbf{p} and \mathbf{L} . This is not the only independent symmetry which can be generated via constructing algebraically dependent quantities from \mathbf{L} and \mathbf{p} . Indeed, the vector $\mathbf{Q} = \mathbf{p} \times \mathbf{L}$ is the analog of the Laplace-Runge-Lenz vector of the Coulomb-Kepler system, which can be treated as a generator of the corresponding canonical symmetry transformations. The integrals \mathbf{L} and \mathbf{Q} , $\mathbf{LQ} = 0$, form together with the Hamiltonian the nonlinear algebra:

$$\{L_i, V_j\} = \epsilon_{ijk} V_k, \quad \{Q_i, Q_j\} = -2H \epsilon_{ijk} L_k, \quad \{H, V_i\} = 0, \quad V_i = L_i, Q_i.$$

The renormalized (at $\mathbf{p} \neq 0$) vector $\mathbf{R} \equiv \mathbf{Q}/\sqrt{\mathbf{p}^2}$ and the angular momentum vector \mathbf{L} form the Lorentz algebra $so(3,1)$. One can construct another vector, $\mathbf{K} \equiv \mathbf{L} \times \mathbf{R} \cdot L^{-1}$, $L = \sqrt{\mathbf{L}^2}$, which together with \mathbf{R} and \mathbf{L} provides us with the complete set of the three orthogonal vectors, $\mathbf{R}\mathbf{L} = \mathbf{K}\mathbf{L} = \mathbf{R}\mathbf{K} = 0$, of the same norm, $\mathbf{R}^2 = \mathbf{K}^2 = \mathbf{L}^2$. They form the following nonlinear algebra:

$$\{L_i, V_j\} = \epsilon_{ijk} V_k, \quad \{R_i, R_j\} = \{K_i, K_j\} = -\epsilon_{ijk} L_k, \quad \{R_i, K_j\} = \delta_{ij} L, \quad (2.14)$$

where $V_i = L_i, R_i, K_i$. One notes that the scalar L rotates the set of vectors \mathbf{R} and \mathbf{K} :

$$\{L, R_i\} = -K_i, \quad \{L, K_i\} = R_i. \quad (2.15)$$

Therefore, the vector integral \mathbf{K} also forms the $so(3,1)$ algebra with the orbital angular momentum and is (non-canonically) conjugate to \mathbf{R} (see the last Poisson bracket relation in Eq. (2.14)). The rotated about \mathbf{L} vectors $\mathbf{R}' = \mathbf{R} \cdot \cos \varphi + \mathbf{K} \cdot \sin \varphi$ and $\mathbf{K}' = \mathbf{K} \cdot \cos \varphi - \mathbf{R} \cdot \sin \varphi$, $\varphi = \text{const}$, possess exactly the same set of properties as \mathbf{R} and \mathbf{K} .

There are also the so called dynamical integrals of motion [8] depending explicitly on time and they appear as follows. Solving the equations of motion $\dot{\mathbf{r}} = \mathbf{p}$, $\dot{\mathbf{p}} = 0$, one gets $\mathbf{r} = \mathbf{p} \cdot (t - t_0) + \mathbf{X}$, $\mathbf{X} \equiv \mathbf{r}(t_0)$. One can treat $\mathbf{X} = \mathbf{r} - \mathbf{p} \cdot (t - t_0)$ as a vector integral of motion dependent explicitly on time: $\frac{d}{dt} \mathbf{X} = \{\mathbf{X}, H\} + \partial \mathbf{X} / \partial t = 0$. It generates the transformations $x_i \rightarrow x_i - \varepsilon_i(t - t_0)$ corresponding to the Galilei boosts. The integrals of motion \mathbf{p} and \mathbf{X} satisfy the Heisenberg algebra $\{X_i, X_j\} = \{p_i, p_j\} = 0$, $\{X_i, p_j\} = 1 \cdot \delta_{ij}$. Algebraically, the vector \mathbf{X} is equivalent to the vector integral not containing the explicit dependence on time, $\mathbf{X}_\perp = \mathbf{r} - \mathbf{p}(\mathbf{pr}) \cdot \mathbf{p}^{-2} = \mathbf{Q} \cdot \mathbf{p}^{-2}$, $\mathbf{X}_\perp \mathbf{p} = 0$, and to the scalar $D = \mathbf{X}\mathbf{p} = \mathbf{rp} - \mathbf{p}^2 \cdot (t - t_0)$ being a dynamical integral of motion generating the time dilations [8, 38]. One notes that the angular momentum \mathbf{L} can also be treated as an integral algebraically dependent on \mathbf{X} and \mathbf{p} : $\mathbf{L} = \mathbf{X} \times \mathbf{p} = \mathbf{X}_\perp \times \mathbf{p}$. On the other hand, from the equations of motion it follows that $\frac{d}{dt} \mathbf{r}^2 = 2\mathbf{pr}$, $\frac{d}{dt}(\mathbf{pr}) = \mathbf{p}^2$. Integration of the second equation gives rise to the dynamical integral D and the subsequent integration of the first equation gives, similar to Eq. (2.10), $\mathbf{r}^2(t) = \mathbf{p}^2 \cdot (t - t_0)^2 + 2D \cdot (t - t_0) + \mathbf{r}^2(t_0)$. The last relation can be rewritten in the form of the dynamical integral of motion $\mathcal{R} = \mathbf{r}^2(t_0) = \mathbf{r}^2 - \mathbf{p}^2 \cdot (t - t_0)^2 - 2D \cdot (t - t_0)$, which generates the time special conformal transformations [8, 38]. It can be represented equivalently as

$$\mathcal{R} = (\mathbf{L}^2 + D^2) \cdot \mathbf{p}^{-2}. \quad (2.16)$$

The integrals D , $2H$ and \mathcal{R} form the same algebra as the scalars \mathbf{pr} , \mathbf{p}^2 and \mathbf{r}^2 (the latter set is reduced to these integrals at the initial moment $t = t_0$), which is the $so(2,1) \sim sl(2, R)$ algebra [41]-[46]:

$$\{\mathcal{J}_0, \mathcal{J}_1\} = \mathcal{J}_2, \quad \{\mathcal{J}_0, \mathcal{J}_2\} = -\mathcal{J}_1, \quad \{\mathcal{J}_1, \mathcal{J}_2\} = -\mathcal{J}_0, \quad (2.17)$$

where $\mathcal{J}_0 = \frac{1}{4}(2H + \mathcal{R})$, $\mathcal{J}_1 = \frac{1}{4}(2H - \mathcal{R})$, $\mathcal{J}_2 = \frac{1}{2}D$, with the Casimir central element $\mathcal{C} = -\mathcal{J}_0^2 + \mathcal{J}_1^2 + \mathcal{J}_2^2 = -\frac{1}{4}\mathbf{L}^2 \leq 0$. The dynamical integral D together with H commutes in the sense of Poisson brackets with the vector integrals \mathbf{L} , \mathbf{R} and \mathbf{K} , whereas the integral \mathcal{R} due to Eqs. (2.15), (2.16) has nontrivial Poisson bracket relations with \mathbf{R} and \mathbf{K} .

Let us turn to the charge-monopole system, where instead of the orbital angular momentum, the vector $\mathbf{J} = \mathbf{L} - \nu \mathbf{n}$ is conserved. Since the equations of motion for the scalar

variables \mathbf{r}^2 and $\mathbf{P}\mathbf{r}$ look exactly as the corresponding equations for the free particle (with the change of \mathbf{p} for \mathbf{P}), the charge-monopole system has the set of the scalar dynamical integrals of motion of the same form [8],

$$D = \mathbf{P}\mathbf{r} - \mathbf{P}^2 \cdot (t - t_0), \quad \mathcal{R} = \mathbf{r}^2 - \mathbf{P}^2 \cdot (t - t_0)^2 - 2D \cdot (t - t_0). \quad (2.18)$$

Though P_i are characterized by the nontrivial Poisson brackets (2.8), the scalar dynamical integrals (2.18) and the Hamiltonian generate the same $so(2,1)$ algebra (2.17) as in a free case with the Casimir central element

$$\mathcal{C} = -\mathcal{J}_0^2 + \mathcal{J}_1^2 + \mathcal{J}_2^2 = -\frac{1}{4}(\mathbf{J}^2 - \nu^2) < 0. \quad (2.19)$$

The free particle's vector integrals of motion \mathbf{R} and \mathbf{K} also have analogs in the charge-monopole system. These are given by the vectors

$$\mathbf{R} \equiv \mathbf{N} \cdot \frac{J^2}{\sqrt{J^2 - \nu^2}}, \quad \mathbf{K} \equiv \mathbf{J} \times \mathbf{N} \cdot \frac{J}{\sqrt{J^2 - \nu^2}}, \quad (2.20)$$

where the vector integral $\mathbf{N} = (\mathbf{n} + \nu J^{-1} \cdot \mathbf{j}) \cdot \cos \tau - \mathbf{j} \times \mathbf{n} \cdot \sin \tau$ can be identified with the vector $\mathbf{n}_\perp(t_0)$ (see Eq. (2.11)). Vector \mathbf{N} satisfies the relations $\mathbf{N}\mathbf{J} = 0$, $\mathbf{N}^2 = 1 - \nu^2 J^{-2}$, and $\{N_i, N_j\} = -\nu^2 J^{-4} \cdot \epsilon_{ijk} J_k$, $\{N_i, \mathbf{r}^2\} = \{N_i, \mathbf{P}\mathbf{r}\} = \{N_i, \mathbf{P}^2\} = 0$. From these relations we find that $\mathbf{R}\mathbf{J} = \mathbf{K}\mathbf{J} = \mathbf{R}\mathbf{K} = 0$, $\mathbf{R}^2 = \mathbf{K}^2 = \mathbf{J}^2$, and that the Poisson bracket algebra of the complete set of orthogonal vectors \mathbf{J} , \mathbf{R} and \mathbf{K} is given by Eq. (2.14) with \mathbf{L} changed for \mathbf{J} . This set of vector integrals is in involution with the dynamical scalar integral of motion D and with H , but the vectors \mathbf{R} and \mathbf{K} have nontrivial Poisson bracket relations with the dynamical integral $\mathcal{R} = (D^2 + \mathbf{J}^2 - \nu^2)/2H$ (see Eq. (2.15)).

Therefore, the dynamical geometry similarity of the charge-monopole system to the 3D free particle discussed in subsection 2.1 also reveals itself in existence of similar sets of integrals of motion (depending and not depending explicitly on time), which form between themselves the same (nonlinear) Lie-Poisson algebras.

2.4 Charge-monopole as a reduced E(3) system

Like a 3D free particle, the charge-monopole system may be treated as a reduced E(3) system. To get such an interpretation, let us pass over from the Hamiltonian variables \mathbf{r} and \mathbf{P} to the set of variables \mathbf{n} , \mathbf{J} , r and $P_r = \mathbf{P}\mathbf{r} \cdot r^{-1}$. They have the following Poisson brackets:

$$\{r, P_r\} = 1, \quad \{r, \mathbf{n}\} = \{r, \mathbf{J}\} = \{P_r, \mathbf{n}\} = \{P_r, \mathbf{J}\} = 0, \quad (2.21)$$

$$\{J_i, J_j\} = \epsilon_{ijk} J_k, \quad \{J_i, n_j\} = \epsilon_{ijk} n_k, \quad \{n_i, n_j\} = 0. \quad (2.22)$$

Poisson brackets (2.22) correspond to the algebra of generators of the Euclidean group E(3) with J_i being a set of generators of rotations and n_i identified as generators of translations. The quantities \mathbf{n}^2 and $\mathbf{J}\mathbf{n}$ lying in the center of $e(3)$ algebra, $\{\mathbf{n}^2, n_i\} = \{\mathbf{n}^2, J_i\} = \{\mathbf{n}\mathbf{J}, n_i\} = \{\mathbf{n}\mathbf{J}, J_i\} = 0$, are fixed in the present case by the relations

$$\mathbf{n}^2 = 1, \quad \mathbf{n}\mathbf{J} = -\nu. \quad (2.23)$$

In terms of the introduced variables, the Hamiltonian of the system takes the form

$$H = \frac{1}{2}P_r^2 + \frac{(\mathbf{J} \times \mathbf{n})^2}{2r^2}. \quad (2.24)$$

Therefore, the charge-monopole system can be treated as the E(3) system reduced by the conditions (2.23) fixing the Casimir elements and supplemented by the independent canonically conjugate variables r and P_r . It is the second relation from Eq. (2.23) that encodes the topological difference between the charge-monopole and the 3D free particle cases: for $\nu \neq 0$, the space given by the spin vector \mathbf{J} is homeomorphic to $\mathbb{R}^3 - \{0\}$, ($J > |\nu|$), whereas for $\nu = 0$ the corresponding space \mathbb{R}^3 is topologically trivial. In the next section we shall discuss the physical consequences of the nontrivial topological structure of the charge-monopole system.

In accordance with Eqs. (2.23), one could treat the vector $\mathbf{L} = \mathbf{J} + \nu\mathbf{n}$, $\mathbf{L}\mathbf{n} = 0$, as an orbital angular momentum. However, the Poisson bracket relations $\{L_i, L_j\} = \epsilon_{ijk}(L_k + \nu n_k)$ following from (2.22) and restriction $\mathbf{L}^2 > 0$ corresponding to $J > |\nu|$ prevent such interpretation. Nevertheless, as we shall see, it is possible to treat \mathbf{L} as an orbital angular momentum in extended physically equivalent formulation of the model.

3 Monopole Chern-Simons term and spin

This section contains mainly the known results which are necessary for the self-contained presentation of the subsequent analysis.

The integrand in action corresponding to the charge-monopole interaction term, $\theta = e\dot{\mathbf{r}}\mathbf{A}(\mathbf{r})dt$, can be treated as a differential one-form, $\theta = e\mathbf{A}(\mathbf{r})d\mathbf{r}$. Then the relations (2.2) defining the monopole vector potential are equivalent to the relation

$$d\theta = \frac{\nu}{2r^3} \cdot \epsilon_{ijk}r_i dr_j \wedge dr_k. \quad (3.1)$$

The right-hand side of Eq. (3.1) is the gauge-invariant curvature two-form,

$$d\theta = e\mathcal{F}, \quad \mathcal{F} = \frac{1}{2}F_{ij}dr_i \wedge dr_j \quad (3.2)$$

and, consequently, the gauge-non-invariant one-form θ has a sense of $(0+1)$ -dimensional CS term [22]-[27]. The two-form (3.1) can be represented equivalently as

$$d\theta = \frac{\nu}{2}\epsilon_{ijk}n_i dn_j \wedge dn_k, \quad n_i = r_i \cdot r^{-1}. \quad (3.3)$$

Via the (local) parametrization by the spherical angles, $\mathbf{n} = \mathbf{n}(\vartheta, \varphi)$, this can be treated as the differential area of a two-sphere multiplied by the charge-monopole coupling constant: $d\theta = \nu d(\cos \vartheta) \wedge d\varphi$. If the vector $\mathbf{n}(t)$ describes some closed curve Γ on the sphere (i.e. if $\mathbf{n}(t_1) = \mathbf{n}(t_2)$)¹, the Stokes theorem gives $\oint_{\Gamma} \theta = \int_{S_+} d\theta = -\int_{S_-} d\theta$, where $\Gamma = \partial S_+ = -\partial S_-$,

¹ According to Eqs. (2.11), (2.12) and relation $(\mathbf{Pr})' = \mathbf{P}^2 = \text{const}$, such a closed curve is smooth.

$S_+ \cup S_- = S^2$. Within the path-integral quantization scheme, the alternative representations for the same charge-monopole interaction term in the action can differ only in $2\pi n$, $n \in \mathbb{Z}$,

$$\int_{S_+} d\theta - \left(- \int_{S_-} d\theta \right) = \nu \int_{S^2} d \cos \vartheta \wedge d\varphi = 4\pi\nu = 2\pi n, \quad (3.4)$$

and we arrive at the Dirac quantization condition for the charge-monopole coupling constant: $2\nu = n \in \mathbb{Z}$. Defining the dependent variables $s_i = -\nu n_i$, one can represent (3.3) equivalently as

$$\omega_{spin} = d\theta = -\frac{1}{2\mathbf{s}^2} \epsilon_{ijk} s_i ds_j \wedge ds_k, \quad s_i s_i = \nu^2. \quad (3.5)$$

The two-form (3.5) is closed and nondegenerate, and can be treated as a symplectic form corresponding to the symplectic potential θ . If we drop out the kinetic term in the charge-monopole action (that corresponds to taking the charge's zero mass limit, $m \rightarrow 0$, [23]), we get the CS action

$$S = \int_{t_1}^{t_2} \theta, \quad (3.6)$$

describing the spin system. Indeed, for any function f on the sphere S^2 , the symplectic form (3.5) defines the Hamiltonian vector field X_f on the cotangent bundle T^*S^2 : $i_{X_f} \omega(Y) = \omega(X_f, Y) = df$. This gives $X_f = \partial_a f \omega^{ab} \frac{\partial}{\partial x^b}$, and defines the corresponding Poisson brackets, $\{f, g\} = -\omega(X_f, X_g)$. Here x^a , $a = 1, 2$, are the local coordinates on S^2 and ω^{ab} are the elements of the matrix inverse to the symplectic matrix ω_{ab} , $\omega_{spin} = \frac{1}{2} \omega_{ab} dx^a \wedge dx^b$. Taking into account Eq. (3.5), we get $\{s_i, s_k\} = \epsilon_{ijk} s_k$, $s_i s_i = \nu^2$. These relations define the classical spin system with fixed spin modulus. Geometric quantization [47, 48] applied to such a system (for the details see ref. [33]) leads to the same Dirac quantization of the parameter ν , $|\nu| = j$, $j = 1/2, 1, 3/2, \dots$, and results in $(2j+1)$ -dimensional representation of $su(2)$,

$$s_1 = \frac{1-z^2}{2} \frac{d}{dz} + jz, \quad s_2 = i \frac{1+z^2}{2} \frac{d}{dz} - ijz, \quad s_3 = z \frac{d}{dz} - j, \quad (3.7)$$

realized on the space of holomorphic functions with the basis $\psi_j^k \propto z^{j+k}$, $k = -j, -j+1, \dots, +j$, $s_3 \psi_j^k = k \psi_j^k$, and scalar product

$$(\psi_1, \psi_2) = \frac{2j+1}{\pi} \int \int \frac{\overline{\psi_1(z)} \psi_2(z)}{(1+|z|^2)^{2j+2}} d^2 z.$$

The classical relation $\mathbf{s}^2 = \nu^2$ is changed for the quantum relation $\mathbf{s}^2 = j(j+1)$. Here the complex variable z is related to the spherical angles via the stereographic projection $z = \tan \frac{\vartheta}{2} e^{i\varphi}$ from the north pole, or via $z = \cot \frac{\vartheta}{2} e^{-i\varphi}$ for the projection from the south pole. In both cases the symplectic two-form is represented as

$$\omega_{spin} = 2i\nu \frac{d\bar{z} \wedge dz}{(1+\bar{z}z)^2}. \quad (3.8)$$

Geometrically, the obtained spin system is a Kähler manifold [27] with Kähler potential $\mathcal{K} = 2i\nu \ln(1+\bar{z}z)$: $\omega_{spin} = \frac{\partial^2}{\partial z \partial \bar{z}} \mathcal{K} \cdot d\bar{z} \wedge dz$, $\bar{z} = z^*$. Locally, in spherical coordinates the spin Lagrangian is given by

$$L_{spin} = \nu \cos \vartheta \dot{\varphi}, \quad (3.9)$$

and in terms of global complex variable the Lagrangian (3.9) takes the form

$$L_{spin} = i\nu \frac{\bar{z}\dot{z} - \dot{\bar{z}}z}{1 + \bar{z}z}. \quad (3.10)$$

The appearance of the two stereographic projections for the spin system (3.6) reflects the above mentioned necessity to work in two charts in the case of the initial ($m \neq 0$) charge-monopole system to escape the problems with Dirac string singularities. In terms of globally defined independent variables z, \bar{z} no gauge invariance left in the spin system given by the Lagrangian (3.10) but it is hidden in a fibre bundle structure reflected, in particular, in the presence of two charts [47].

We conclude that the charge-monopole interaction term has a nature of (0+1)-dimensional Abelian CS term, that leads to the Dirac quantization of the charge-monopole coupling constant, $|\nu| = j$. In the limit of zero charge's mass the total charge-monopole Lagrangian is reduced to the Lagrangian (3.10) in terms of independent variables z and \bar{z} , which describes the spin- j system. In what follows, we consider other possibilities to describe spin system proceeding from its nature associated with the monopole CS term.

4 Charge-monopole as a particle with spin

The spin nature of the charge-monopole CS term and observed free character of the charge's dynamics allow us to get the alternative description for the charge-monopole system as a free particle of fixed spin with translational and spin degrees of freedom interacting via the helicity constraint. To find such a description, we forget for the moment that the spin system has been obtained via the identification $s_i = -\nu n_i$, (in the limit $m \rightarrow 0$), and simply start with its Hamiltonian description given by the symplectic form (3.5) and corresponding brackets $\{s_i, s_k\} = \epsilon_{ijk}s_k$. The canonical Hamiltonian of the spin theory given by the first order action (3.6) or by the Lagrangian (3.10) is equal to zero. Let us extend such a pure spin system by adding to it independent translational degrees of freedom described by the particle's coordinates r_i and canonically conjugate momenta p_i , i.e. we suppose that the corresponding symplectic two-form is

$$\omega = dp_i \wedge dr_i + \omega_{spin}. \quad (4.1)$$

Moreover, let the dynamics of the system is given by the Hamiltonian $H = \frac{1}{2}\mathbf{p}^2$. Then this Hamiltonian and relations (3.5), (4.1) specify the nonrelativistic free particle with internal degrees of freedom describing spin of fixed value. In such a system we have 6+2 independent phase space variables instead of 6 variables in the initial charge-monopole system. We can reduce such an extended system to the initial system by introducing into it one first class constraint. Having in mind the identification $s_i = -\nu n_i$, for the initial system (2.1), let us postulate the helicity constraint

$$\chi \equiv \mathbf{s} \cdot \mathbf{r} + \nu r \approx 0. \quad (4.2)$$

This constraint can be interpreted as a constraint introducing interaction between translational and spin degrees of freedom. But the constraint (4.2) is not conserved by the Hamiltonian, $\frac{1}{2}\{\mathbf{p}^2, \chi\} \neq 0$, and for consistency of such a theory we have to modify the latter. For

the purpose, let us find the complete set of gauge-invariant variables commuting in the sense of Poisson brackets with constraint (4.2). Since we have 6 + 2 phase space variables and one constraint, there are 6 independent gauge-invariant variables. They are r_i and

$$\Pi_i \equiv p_i - \frac{1}{r^2} \epsilon_{ijk} r_j s_k, \quad (4.3)$$

$\{r_i, \chi\} = 0$, $\{\Pi_i, \chi\} \approx 0$. Therefore, it is natural to change $H = \frac{1}{2} \mathbf{p}^2$ for its gauge-invariant analog, $H = \frac{1}{2} \mathbf{\Pi}^2$, and to take the sum of it and constraint (4.2) multiplied by an arbitrary function $\lambda = \lambda(t)$,

$$H = \frac{1}{2} \mathbf{\Pi}^2 + \lambda \cdot (\mathbf{s}r + \nu r), \quad (4.4)$$

as a total Hamiltonian [49]. The Poisson brackets for the gauge-invariant variables are

$$\{\Pi_i, \Pi_j\} = -\frac{\mathbf{r}\mathbf{s}}{r^4} \epsilon_{ijk} r_k, \quad \{\Pi_i, r_j\} = \delta_{ij}, \quad \{r_i, r_j\} = 0. \quad (4.5)$$

Taking into account constraint (4.2), the Poisson brackets between Π_i and Π_j coincide with the Poisson brackets (2.8) between P_i and P_j . Identifying Π_i with P_i , the dynamics generated by the Hamiltonian (4.4) for the gauge-invariant variables r_i , Π_i is exactly the same as the dynamics in the initial charge-monopole system (2.1), and the physical content of the extended system (4.4) is the same as that of the initial system (2.1). In particular, the vector

$$\mathbf{J} = \mathbf{r} \times \mathbf{p} + \mathbf{s} \quad (4.6)$$

is the integral of motion, whose components satisfy the Poisson bracket relations $\{J_i, J_j\} = \epsilon_{ijk} J_k$, and generate rotations. It can be represented equivalently as $\mathbf{J} = \mathbf{r} \times \mathbf{\Pi} + (\mathbf{n}\mathbf{s}) \cdot \mathbf{n}$, and on the constraint surface (4.2) takes the form $\mathbf{J} \approx \mathbf{r} \times \mathbf{\Pi} - \nu \cdot \mathbf{n}$, which coincides with (2.4) with identified Π_i and P_i . With the help of relation (4.6), the Hamiltonian (4.4) can be written down equivalently (cf. with Eq. (2.24)),

$$H = \frac{1}{2} p_r^2 + \frac{(\mathbf{J} \times \mathbf{n})^2}{2r^2} + \lambda \cdot (\mathbf{J}\mathbf{n} + \nu), \quad (4.7)$$

where $p_r = \mathbf{p} \cdot \mathbf{r}^{-1}$ is the momentum canonically conjugate to the radial variable r . Since $\mathbf{L}\mathbf{n} = 0$, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, the difference of the present Hamiltonian interpretation of the charge-monopole system as a particle with spin from the reduced E(3) system from section 2.4 is that here the helicity is fixed weakly by the constraint (4.2), whereas there it was fixed strongly by the second relation from Eq. (2.23). As a consequence, here the components of the angular momentum vector \mathbf{L} form the $so(3)$ algebra, $\{L_i, L_j\} = \epsilon_{ijk} L_k$, but they, unlike the total angular momentum vector \mathbf{J} , are not physical variables: $\{L_i, \chi\} \neq 0$. In the present interpretation, we have additional spin variables s_i restricted by the condition $\mathbf{s}^2 = \nu^2$, but the only physical observable constructed from them is the combination $\mathbf{s}\mathbf{n}$, which is fixed by the helicity constraint. The important comment is in order here. Strictly speaking, in correspondence with the discussion above on the necessity of restriction $J^2 > \nu^2$, we have to suppose that the relation $\mathbf{s}^2 = \nu^2$ has to be treated only in the sense of the limit $\mathbf{s}^2 = \nu^2 + \varepsilon^2$, $\varepsilon \rightarrow 0$. This will not only correspond to the specified restriction on J^2 in accordance with relations (4.2) and (4.6), but is necessary for the consistent treatment of the extended model

as a constrained system. Indeed, in correspondence with general theory of gauge systems [49], only in this case the constraint (4.2) can be treated as a good constraint condition, which in the two-dimensional spin phase subspace given by s_i , $\{s_i, s_j\} = \epsilon_{ijk}s_k$, $\mathbf{s}^2 = \nu^2 + \varepsilon^2$, specifies one-dimensional physical subspace (on which it will act transitively).

The Lagrangian corresponding to the described extended system is

$$L = \frac{1}{2}\dot{\mathbf{r}}^2 - \frac{1}{r^2}(\mathbf{r} \times \dot{\mathbf{r}}) \cdot \mathbf{s} - \lambda \cdot (\mathbf{r}\mathbf{s} + \nu r) + L_{spin}, \quad (4.8)$$

with L_{spin} chosen in the form (3.10) and λ treated as a Lagrange multiplier.

We conclude that the system (4.8) describing the nonrelativistic particle of spin ν with interacting translational and spin degrees of freedom (see the second $\mathbf{L}\mathbf{s}$ -coupling term and the third constraint term in Lagrangian) is classically equivalent to the charge-monopole system (2.1).

The quantum theory of the charge-monopole system in such interpretation is obvious. As we have seen, the quantization of the spin variables results in the Dirac condition, $\nu = \epsilon j$, $\epsilon = +$ or $-$, and gives rise to the corresponding $(2j + 1)$ -dimensional representation of the $su(2)$ with spin operators (3.7) acting on the space of holomorphic functions. Choosing the representation diagonal in r_i and realizing p_j as differential operators, $p_j = -i\partial/\partial r_j$, one can work on the space of functions of the form $\Psi^j(\mathbf{r}, z) = \sum_{k=-j}^j \psi_k(\mathbf{r})z^{j+k}$. The quantum analog of the classical constraint takes the form of the quantum condition separating the physical subspace, $(\mathbf{sn} + \epsilon j)\Psi_{phys}^j(\mathbf{r}, z) = 0$. It is necessary to note that quantum mechanically the above mentioned necessity of the regularization $\mathbf{s}^2 = \nu^2 + \varepsilon^2$ is taken into account automatically. Indeed, since the eigenvalues of the operators \mathbf{s}^2 and $(\mathbf{sn})^2$ are separated in a necessary way, $\mathbf{s}^2 = j(j + 1)$, $(\mathbf{sn})^2 = j^2$, one can say that the quantization ‘‘cures’’ the classical system.

5 Charge-monopole system as a symmetric top

In this section we show that the charge-monopole system can also be interpreted as a reduced symmetric spinning top with dynamical tensor of inertia and ‘‘isospin’’ $U(1)$ gauge symmetry. To this end, we return to the spin symplectic form (3.3), and supplement the unit vector $\mathbf{n} \equiv \mathbf{e}_3$ with the two vectors \mathbf{e}_1 and \mathbf{e}_2 forming together the oriented orthonormal set of vectors,

$$\mathbf{e}_a \mathbf{e}_b = \delta_{ab}, \quad \mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3. \quad (5.1)$$

As a consequence of basic relations (5.1), the vectors \mathbf{e}_a , $a = 1, 2, 3$, satisfy also the completeness relation

$$e_a^i e_a^j = \delta^{ij}. \quad (5.2)$$

Using Eqs. (5.1), (5.2), one can represent the two-form (3.3) as

$$d\theta = \nu d\mathbf{e}_1 \wedge d\mathbf{e}_2, \quad (5.3)$$

from which we get another representation for the one-form,

$$\theta = \frac{\nu}{2}(\mathbf{e}_1 d\mathbf{e}_2 - \mathbf{e}_2 d\mathbf{e}_1). \quad (5.4)$$

Taking into account the relation $\dot{\mathbf{e}}_3^2 = (\dot{\mathbf{e}}_1 \mathbf{e}_3)^2 + (\dot{\mathbf{e}}_2 \mathbf{e}_3)^2$, we get the alternative Lagrangian for the charge-monopole system,

$$L = \frac{1}{2}r^2 + \frac{1}{2}r^2 \left((\dot{\mathbf{e}}_1 \cdot \mathbf{e}_1 \times \mathbf{e}_2)^2 + (\dot{\mathbf{e}}_2 \cdot \mathbf{e}_1 \times \mathbf{e}_2)^2 \right) + \frac{\nu}{2}(\mathbf{e}_1 \dot{\mathbf{e}}_2 - \mathbf{e}_2 \dot{\mathbf{e}}_1) - \frac{\lambda_1}{2}(\mathbf{e}_1^2 - 1) - \frac{\lambda_2}{2}(\mathbf{e}_2^2 - 1) - \lambda_{12}\mathbf{e}_1\mathbf{e}_2, \quad (5.5)$$

with λ_1 , λ_2 and λ_{12} being Lagrange multipliers, variation over which gives the Lagrangian constraints $\mathbf{e}_1^2 - 1 = 0$, $\mathbf{e}_2^2 - 1 = 0$ and $\mathbf{e}_1\mathbf{e}_2 = 0$. With these constraints, the equations of motion for $\mathbf{r} = r\mathbf{e}_3$, $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$, following from (5.5) coincide with the charge-monopole Lagrange equations (2.3). To prove the complete equivalence of the model (5.5) to the initial system (2.1), we pass over to the Hamiltonian formalism. The canonical Hamiltonian corresponding to Lagrangian (5.5) is

$$H_c = \frac{1}{2}p_r^2 + \frac{1}{2r^2}(\mathbf{J} \times \mathbf{e}_3)^2 + \frac{\lambda_1}{2}(\mathbf{e}_1^2 - 1) + \frac{\lambda_2}{2}(\mathbf{e}_2^2 - 1) + \lambda_{12}\mathbf{e}_1\mathbf{e}_2, \quad (5.6)$$

where $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$ and the total angular momentum is $\mathbf{J} = \mathbf{e}_1 \times \mathbf{p}_1 + \mathbf{e}_2 \times \mathbf{p}_2$ with $\mathbf{p}_{1,2}$ being the momenta canonically conjugate to $\mathbf{e}_{1,2}$. Note that in terms of the velocity phase space variables the total angular momentum is reduced to $\mathbf{J} = r^2 \cdot \mathbf{e}_3 \times \dot{\mathbf{e}}_3 - \nu \mathbf{e}_3$ in correspondence with Eqs. (2.4). The application of Dirac-Bergmann theory to the system (5.5) results in the following complete set of constraints:

$$\pi_1 \approx 0, \quad \pi_2 \approx 0, \quad \pi_{12} \approx 0, \quad (5.7)$$

$$\mathbf{e}_1^2 - 1 \approx 0, \quad \mathbf{p}_1\mathbf{e}_1 \approx 0, \quad \mathbf{e}_2^2 - 1 \approx 0, \quad \mathbf{p}_2\mathbf{e}_2 \approx 0, \quad \mathbf{e}_1\mathbf{e}_2 \approx 0, \quad \mathbf{p}_1\mathbf{e}_2 + \mathbf{p}_2\mathbf{e}_1 \approx 0, \quad (5.8)$$

$$\mathbf{p}_1\mathbf{e}_2 - \mathbf{p}_2\mathbf{e}_1 + \nu \approx 0, \quad (5.9)$$

with the momenta π_1 , π_2 and π_{12} canonically conjugate to Lagrange multipliers. Constraints (5.7) mean that the Lagrange multipliers are pure gauge variables and can be completely excluded, e.g., by supplementing (5.7) with the gauge conditions $\lambda_1 \approx 0$, $\lambda_2 \approx 0$, $\lambda_{12} \approx 0$. The constraints (5.8) form the subset of second class constraints, whereas (5.9) is the first class constraint. Reduction of the symplectic two-form of the system, $\omega = d\mathbf{p}_1 \wedge d\mathbf{e}_1 + d\mathbf{p}_2 \wedge d\mathbf{e}_2 + dp_r \wedge dr$, to the surface of second class constraints (5.8) is given by

$$\omega = \frac{1}{2}d(\mathbf{J} \times \mathbf{e}_a) \wedge d\mathbf{e}_a + dp_r \wedge dr, \quad (5.10)$$

where the summation over $a = 1, 2, 3$ is assumed. The two-form (5.10) is the symplectic form on the reduced phase space which is described by the basis of vectors \mathbf{e}_a , $a = 1, 2, 3$, subject to conditions (5.1) as strong relations, by the angular momentum vector \mathbf{J} and by the canonically conjugate radial variables r and p_r . From (5.10) we obtain the following Poisson-Dirac brackets on the reduced phase space:

$$\{e_a^i, e_b^j\} = 0, \quad \{J^i, e_a^j\} = \epsilon^{ijk} e_a^k, \quad \{J^i, J^j\} = \epsilon^{ijk} J^k, \quad \{r, p_r\} = 1, \quad (5.11)$$

and all other brackets for radial variables r and p_r to be equal to zero. The remaining constraint (5.9) takes the form

$$\chi = \mathbf{J}\mathbf{e}_3 + \nu \approx 0, \quad (5.12)$$

and the total Hamiltonian of the system is reduced to the canonical Hamiltonian extended by the first class constraint multiplied by an arbitrary function $\lambda(t)$:

$$H = \frac{1}{2}p_r^2 + \frac{1}{2r^2}(\mathbf{J} \times \mathbf{e}_3)^2 + \lambda \cdot (\mathbf{J}\mathbf{e}_3 + \nu). \quad (5.13)$$

To understand the physical sense of the obtained system, let us define the scalar quantities $I_a \equiv -\mathbf{J}\mathbf{e}_a$, $I_a I_a = \mathbf{J}^2$. They satisfy the following Poisson bracket relations:

$$\{I_a, I_b\} = \epsilon_{abc} I_c, \quad \{I_a, e_b^i\} = \epsilon_{abc} e_c^i, \quad \{I_a, J^i\} = 0. \quad (5.14)$$

Since I_a commute with J_i and satisfy $su(2)$ algebra, one can treat them as components of the isospin vector. On the other hand, due to the basic relations (5.1), the quantities e_a^i can be treated as the elements of the group $SO(3)$. Then the quantities J^i have a sense of the basis of the left-invariant vector fields on this group whereas the quantities $-I_a$ can be identified with the basis of the right-invariant vector fields [27, 50]. Due to the relations (5.14), the first class constraint (5.12) generates gauge transformations of the system which have a sense of $SO(2) \sim U(1)$ isospin rotations generated by I_3 . Under such transformations the vectors \mathbf{J} and \mathbf{e}_3 are invariant. The electromagnetic $U(1)$ gauge invariance of the curvature form (3.2) corresponding to the CS charge-monopole coupling term is changed here for the described $SO(2) \sim U(1)$ gauge invariance of the two-form (5.3) corresponding to the CS form (5.4). Since $(\mathbf{J} \times \mathbf{e}_3)^2 = I_1^2 + I_2^2$, the system given by the Hamiltonian (5.13) and brackets (5.11) can be identified as a symmetric spinning top with the symmetry axis given by \mathbf{e}_3 and dynamical moment of inertia $\mathcal{I}_1 = \mathcal{I}_2 = r^2$. The isospin $U(1)$ gauge symmetry generated by the constraint (5.12) means that the rotation about the symmetry axis \mathbf{e}_3 is of pure gauge nature and, so, is not observable.

The formal difference of the symmetric spinning top system from the reduced E(3) system discussed in section 2.4 is that here the condition $\mathbf{J}\mathbf{n} = \mathbf{J}\mathbf{e}_3 = -\nu$ appears in the form of first class constraint (weak equality) unlike the strong equality in the case of the reduced E(3) system. The physical content of both systems is exactly the same. Indeed, besides the radial variables r and p_r , the only observable quantities (commuting in the sense of Dirac-Poisson brackets with the constraint (5.12)) are the vector of angular momentum \mathbf{J} and the unit vector $\mathbf{e}_3 \equiv \mathbf{n}$. So, the E(3) system can also be treated as a symmetric spinning top reduced by the action of the first class constraint (5.12).

The system is quantized as follows. In correspondence with the classical relations (5.1), (5.2) and the sense of the complete orthonormal set of vectors \mathbf{e}_a , they can be parametrized by the three Euler angles α, β, γ , and we can choose a representation diagonal in these angle variables and in the radial variable r . Then the components of the operator \mathbf{J} are realized in the form of linear differential operators of angular variables [51, 52]. An arbitrary state can be decomposed over the complete basis of Wigner functions,

$$\Psi(r, \alpha, \beta, \gamma) = \sum_{j,s,k} \psi_{j,s,k}(r) D_{s,k}^j(\alpha, \beta, \gamma),$$

where j takes either integer, $j = 0, 1, 2, \dots$, or half-integer, $j = 1/2, 3/2, \dots$, values, and $s, k = -j, -j+1, \dots, +j$, $\mathbf{J}^2 D_{s,k}^j = j(j+1) D_{s,k}^j$, $J_3 D_{s,k}^j = s D_{s,k}^j$, $I_3 D_{s,k}^j = k D_{s,k}^j$. The quantum

analog of the first class constraint (5.12) is transformed into the equation separating the physical states:

$$(I_3 - \nu)\Psi_{phys} = 0. \quad (5.15)$$

This equation has nontrivial solutions only for $\nu = n/2$, $n \in \mathbb{Z}$, i.e. we arrive once again at the Dirac quantization condition, and the corresponding solutions of Eq. (5.15) have the form

$$\Psi_{phys}(r, \alpha, \beta, \gamma) = \sum_{j,s} ' \psi_{s, \frac{n}{2}}^j(r) D_{s, \frac{n}{2}}^j(\alpha, \beta, \gamma), \quad (5.16)$$

where prime means that in the sum j takes the values $j = |n/2|, |n/2|+1, \dots$. Once again, let us note that here the quantization separates in the necessary way the value of the quantized parameter $|\nu| = |\frac{n}{2}|$ and possible eigenvalues of the angular momentum operator in the physical subspace: $\mathbf{J}^2 = j(j+1) > \nu^2$.

It is interesting to note that in the present “spinning top” picture for the charge-monopole system the *total* angular momentum is represented in terms of the isospin angular momentum as $\mathbf{J} = -\mathbf{e}_a I_a$, whereas from the point of view of the field theoretical mechanism [2, 3], only *spin* part of the total angular momentum vector of the charge-monopole system is created from the isospin degrees of freedom. The seeming contradiction is explained by the constraint (5.15) prescribing the total angular momentum to take the values starting from the minimal value $|n/2| = |\nu|$, which can be interpreted as the “internal” spin of the charge-monopole system, and in this sense here spin is also created by isospin.

6 Higher-derivative form of CS term

The charge-monopole system can also be described by the Lagrangian with CS term represented in a higher-derivative form. To show this, we write down the CS form (5.4) as $\theta = -\nu \mathbf{e}_2 d\mathbf{e}_1$. Identifying the vectors \mathbf{e}_3 and \mathbf{e}_1 with the unit vectors $\mathbf{n} = \mathbf{r} \cdot r^{-1}$ and $\dot{\mathbf{n}} \cdot |\dot{\mathbf{n}}|^{-1}$, respectively, and taking into account that $\mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_1$, the charge-monopole coupling term can be written down in the equivalent higher-derivative form

$$L_{int} = -\frac{\nu}{\dot{\mathbf{n}}^2} \mathbf{n} \cdot (\dot{\mathbf{n}} \times \ddot{\mathbf{n}}). \quad (6.1)$$

Simple geometrical consideration shows that the higher derivative term $(\mathbf{n} \times \dot{\mathbf{n}}) \cdot \ddot{\mathbf{n}}/\dot{\mathbf{n}}^2$ has a sense of angular velocity of rotation of the vector $\dot{\mathbf{n}}$ about the vector \mathbf{n} . The total charge-monopole Lagrangian is given by

$$L = \frac{1}{2} \dot{\mathbf{r}}^2 - \nu \frac{r}{(\mathbf{r} \times \dot{\mathbf{r}})^2} (\mathbf{r} \times \dot{\mathbf{r}}) \cdot \ddot{\mathbf{r}}. \quad (6.2)$$

Naively, the equations of motion following from Lagrangian (6.2),

$$\frac{\partial L}{\partial \mathbf{r}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{r}}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{\mathbf{r}}} = 0, \quad (6.3)$$

are the third order differential equations and their equivalence to the second-order equations (2.3) is not obvious. Let us prove the equivalence of equations of motion (6.3) to Eq. (2.3).

First we note that the higher-derivative term (6.1) satisfies the following relations:

$$\begin{aligned} \mathbf{r} \cdot \partial L_{int} / \partial \mathbf{r} &= 0, & \dot{\mathbf{r}} \cdot \partial L_{int} / \partial \dot{\mathbf{r}} &= -L_{int}, & \ddot{\mathbf{r}} \cdot \partial L_{int} / \partial \ddot{\mathbf{r}} &= +L_{int}, \\ \dot{\mathbf{r}} \cdot \partial L_{int} / \partial \mathbf{r} &= (\mathbf{r}\dot{\mathbf{r}})r^{-2} \cdot L_{int}, & \mathbf{r} \cdot \partial L_{int} / \partial \dot{\mathbf{r}} &= \mathbf{r} \cdot \partial L_{int} / \partial \ddot{\mathbf{r}} = \dot{\mathbf{r}} \cdot \partial L_{int} / \ddot{\mathbf{r}} = 0. \end{aligned} \quad (6.4)$$

Multiplying the equations (6.3) subsequently by the vectors \mathbf{r} , $\dot{\mathbf{r}}$ and $\mathbf{r} \times \dot{\mathbf{r}}$, and using Eqs. (6.4), we arrive at the three equations $\ddot{\mathbf{r}}\mathbf{r} = 0$, $\ddot{\mathbf{r}}\dot{\mathbf{r}} = 0$, $\ddot{\mathbf{r}} \cdot (\mathbf{r} \times \dot{\mathbf{r}}) = -\frac{\nu}{r^3}(\mathbf{r} \times \dot{\mathbf{r}})^2$. Since \mathbf{r} , $\dot{\mathbf{r}} - \mathbf{r}(\mathbf{r}\dot{\mathbf{r}}) \cdot r^{-2}$ and $\mathbf{r} \times \dot{\mathbf{r}}$ form the complete basis of orthogonal vectors, we conclude that the equations of motion (6.3) are equivalent to the obtained system of three scalar equations, which, in turn, is equivalent to one vector equation (2.3). Therefore, the coupling of the charge to the magnetic monopole can be alternatively described by the scale invariant higher derivative term (6.1).

It is worth noting that analogously to the present system, the relativistic massive particle in (2+1) dimensions coupled to the external constant homogeneous electromagnetic field turns out to be classically equivalent [30] to the higher derivative model of relativistic particle with torsion given by the action [31, 33] $S_{tor} = -\int(m + \alpha\kappa)ds$, where $ds^2 = -dx_\mu dx^\mu$, κ is a scale invariant torsion of the particle's world trajectory, $\kappa = \epsilon^{\mu\nu\lambda}x'_\mu x''_\nu x'''_\lambda / (x'')^2$, $x'_\mu = dx_\mu/ds$, and α is a parameter. Such a model underlies (2+1)-dimensional anyons [32, 33], and in the last section we shall discuss the relationship between the charge-monopole system and anyons.

7 Twistor description of the charge-monopole system

The helicity constraint appearing in the charge-monopole system interpreted as a particle with spin is analogous to the helicity-fixing constraint in the case of massless particle with spin. The latter system admits, in particular, the twistor description [53]-[56]. Using this observation, one can get the twistor formulation for the charge-monopole system.

In the twistor approach for the massless particle, the corresponding energy-momentum vector is treated as a “composite” object constructed from the twistor (even spinor) variables, and the helicity fixing constraint generates the U(1) transformations for twistors. By analogy, let us introduce mutually conjugate even complex variables z_a and $\bar{z}_a = z_a^*$, $a = 1, 2$, forming two conjugate spinors. With them, we represent the charge coordinate vector \mathbf{r} as a composite vector,

$$\varphi_i \equiv r_i - z\sigma_i\bar{z} \approx 0, \quad (7.1)$$

where σ_i is the set of Pauli matrices. Introducing the momenta $\mathcal{P}_a, \bar{\mathcal{P}}_a$ canonically conjugate to the spinor variables, $\{z_a, \mathcal{P}_b\} = \delta_{ab}$, $\{\bar{z}_a, \bar{\mathcal{P}}_b\} = \delta_{ab}$, we construct the generators of rotations (the total angular momentum vector),

$$J_i = (\mathbf{r} \times \mathbf{p})_i + \frac{i}{2} (z\sigma_i\mathcal{P} - \bar{\mathcal{P}}\sigma_i\bar{z}), \quad (7.2)$$

forming the su(2) algebra: $\{J_i, J_j\} = \epsilon_{ijk}J_k$. Here \mathbf{p} is the vector canonically conjugate to \mathbf{r} . Following the twistor approach, we also introduce the constraint

$$\chi \equiv \frac{i}{2} (\bar{z}\bar{\mathcal{P}} - z\mathcal{P}) - \nu \approx 0, \quad (7.3)$$

which is in involution with the constraints (7.1) and generates the U(1) gauge transformations for the spinor variables, $z_a \rightarrow z'_a = e^{i\gamma} z_a$, $\bar{z}_a \rightarrow \bar{z}'_a = e^{-i\gamma} \bar{z}_a$, $\mathcal{P}_a \rightarrow \mathcal{P}'_a = e^{-i\gamma} \mathcal{P}_a$, $\bar{\mathcal{P}}_a \rightarrow \bar{\mathcal{P}}'_a = e^{i\gamma} \bar{\mathcal{P}}_a$, with $\gamma = \gamma(t)$ being a parameter of transformation. Taking into account the identity

$$\sigma_{ab}^i \sigma_{cd}^i = \delta_{ab} \delta_{cd} - 2\epsilon_{ac} \epsilon_{bd}, \quad (7.4)$$

with $\epsilon_{ab} = -\epsilon_{ba}$, $\epsilon_{12} = 1$, we get the relation $r \approx \bar{z}z$ as the consequence of the constraint (7.1). This relation and Eq. (7.2) allow us to represent the constraint (7.3) in the equivalent form of the helicity-fixing constraint,

$$\tilde{\chi} \equiv \mathbf{J}\mathbf{r} + \nu r \approx 0. \quad (7.5)$$

Since the total phase space described by r_i , p_i , z_a , \bar{z}_a , \mathcal{P}_a and $\bar{\mathcal{P}}_a$ is 14-dimensional, and we have 4 first class constraints (7.1) and (7.3), there are only 6 independent physical phase space degrees of freedom like in the charge-monopole system. They are given by the observables having zero Poisson brackets with all the constraints. Such independent variables are r_i and

$$\Pi_i \equiv p_i + \frac{1}{2\bar{z}z}(z\sigma_i\mathcal{P} + \bar{\mathcal{P}}\sigma_i\bar{z}), \quad (7.6)$$

for which we have the Poisson bracket relations $\{r_i, r_j\} = 0$, $\{r_i, \Pi_j\} = \delta_{ij}$ and

$$\{\Pi_i, \Pi_j\} = (\bar{z}z)^{-2} \epsilon_{ijk}(\mathbf{r} \times \mathbf{\Pi} - \mathbf{J})_k \approx \frac{\nu}{r^3} \epsilon_{ijk} r_k. \quad (7.7)$$

The weak equality means the equality on the surface of constraints (7.1) and (7.3). Physical variables Π_i correspond here to the charge-monopole variables P_i . The Hamiltonian of the charge-monopole system in the twistor formulation can be taken as the linear combination of $\mathbf{\Pi}^2$ and of the first class constraints,

$$H = \frac{1}{2} \mathbf{\Pi}^2 + \rho_i \cdot \varphi_i + \lambda \cdot \chi, \quad (7.8)$$

with $\rho_i = \rho_i(t)$, $\lambda = \lambda(t)$ being arbitrary functions (Lagrange multipliers). Direct calculation shows that the equations of motion generated by this Hamiltonian are reduced to $\ddot{\mathbf{r}} \approx -\nu r^{-3} \cdot \mathbf{r} \times \dot{\mathbf{r}}$, i.e. the Hamiltonian (7.8) and constraints (7.1), (7.3) give the alternative twistor description for the charge-monopole system.

The quantum theory of the system in the twistor approach is the following. It is natural to choose the representation diagonal in \mathbf{r} , z_a and \bar{z}_a , and realize the canonically conjugate momenta in the form of differential operators. In accordance with constraint (7.1), the physical states have to be of the form $\Psi_{phys} = \delta^{(3)}(\mathbf{r} - z\boldsymbol{\sigma}\bar{z}) \cdot \psi(z, \bar{z})$. Then like in the model of massless particle with spin [54], the quantum analog of the constraint (7.3),

$$\left(\frac{1}{2} \left(\bar{z}_a \frac{\partial}{\partial \bar{z}_a} - z_a \frac{\partial}{\partial z_a} \right) - \nu \right) \Psi_{phys} = 0, \quad (7.9)$$

and the requirement of single-valuedness of the wave functions results in the quantization of the charge-monopole constant, $2\nu = n$, $n \in \mathbb{Z}$. Finally, the physical states subject to Eq. (7.9) will be described by the wave functions of the form

$$\Psi_{phys} = \delta^{(3)}(\mathbf{r} - z\boldsymbol{\sigma}\bar{z}) \cdot \psi_{phys}(z, \bar{z}), \quad \psi_{phys}(z, \bar{z}) = \sum_{k=-\infty}^{\infty} \sum_{a,b=1,2} C_k^{ab} \cdot (z_a)^k (\bar{z}_b)^{n+k}, \quad (7.10)$$

where C_k^{ab} are constants. The action of quantum analogs of classical observables \mathbf{r} and $\mathbf{\Pi}$ on physical wave functions (7.10) is reduced to

$$\begin{aligned} \mathbf{r}\Psi_{phys} &= \delta^{(3)}(\mathbf{r} - z\boldsymbol{\sigma}\bar{z}) \cdot z\boldsymbol{\sigma}\bar{z} \cdot \psi_{phys}(z, \bar{z}), \\ \mathbf{\Pi}\Psi_{phys} &= -i\delta^{(3)}(\mathbf{r} - z\boldsymbol{\sigma}\bar{z}) \cdot (2\bar{z}z)^{-1} \left(z\boldsymbol{\sigma} \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \boldsymbol{\sigma} \bar{z} \right) \cdot \psi_{phys}(z, \bar{z}). \end{aligned} \quad (7.11)$$

By inverse Legendre transformation, analogously to the case of the massless particle with spin [54], one can construct the Lagrangian corresponding to Hamiltonian (7.8). Instead of realizing such constructions, we note here that the list of independent observable quantities and the action of their quantum analogs on the physical states indicate on the possibility to exclude \mathbf{r} and \mathbf{p} as independent variables and to construct the theory realized only in terms of spinor variables. To get the corresponding twistor form of the Lagrangian, it is convenient to start from the higher-derivative Lagrangian (6.2) by putting in it $\mathbf{r} = z\boldsymbol{\sigma}\bar{z}$. The remarkable feature of such a twistor representation is that its substitution into the CS term of the higher derivative form (6.1) reduces the latter to $-i\nu(\dot{z}z - \bar{z}\dot{\bar{z}})/\bar{z}z$, and we arrive at the Lagrangian

$$L = \frac{1}{2} \left(\frac{d}{dt}(z\boldsymbol{\sigma}\bar{z}) \right)^2 + i\nu \frac{\bar{z}\dot{z} - \dot{\bar{z}}z}{\bar{z}z}. \quad (7.12)$$

Let us show that (7.12) describes the charge-monopole system. From the definition of the canonical momenta $\mathcal{P}_a = \partial L / \partial \dot{z}_a$, $\bar{\mathcal{P}}_a = \partial L / \partial \dot{\bar{z}}_a$, we find that the constraint (7.3) is the primary constraint for the system (7.12) and the total Hamiltonian is

$$H = \frac{1}{2z\bar{z}} \pi \bar{\pi} + \lambda \cdot \chi, \quad (7.13)$$

where we have introduced the notation $\pi_a = \mathcal{P}_a - i\nu \bar{z}_a \cdot (\bar{z}z)^{-1}$, $\bar{\pi}_a = \bar{\mathcal{P}}_a + i\nu z_a \cdot (\bar{z}z)^{-1}$. The constraint (7.3) is conserved by the Hamiltonian and, as a consequence, there are no secondary constraints. Therefore, the constraint (7.3) is the first class constraint and the number of physical phase space degrees of freedom is equal to $8 - 2 = 6$. The corresponding 6 independent observables weakly commuting in the Poisson bracket sense with the constraint (7.3) are

$$r_i = z\sigma_i\bar{z}, \quad \Pi_i = \frac{1}{2z\bar{z}} (\bar{\mathcal{P}}\sigma_i\bar{z} + z\sigma_i\mathcal{P}). \quad (7.14)$$

These quantities satisfy the following Poisson bracket relations: $\{r_i, r_j\} = 0$, $\{r_i, \Pi_j\} = \delta_{ij}$, $\{\Pi_i, \Pi_j\} \approx \nu r^{-3} \epsilon_{ijk} r_k$, where the weak equality means the equality on the surface of the constraint (7.3). Obviously, they have the sense of the charge-monopole variables r_i and P_i represented in the composite form (7.14) in terms of twistor variables. The variables Π_i are nothing else as the classical analogs of the quantum operators corresponding to observables (7.6), whose action is reduced to the surface of the constraints (7.1), i.e. they are the classical analogs of operators (7.11). The conserving $su(2)$ generators are represented here in the form $\mathbf{J} = \mathbf{r} \times \mathbf{\Pi} - \nu \mathbf{n}$ with $\mathbf{n} = \mathbf{r} \cdot r^{-1}$ and r_i, Π_i given by Eq. (7.14). Direct verification shows that the Hamiltonian (7.13) generates the correct equations of motion, $\ddot{\mathbf{r}} = -\nu r^{-3} \mathbf{r} \times \dot{\mathbf{r}}$. It is worth noting that the system (7.13) can be treated as a reduction of the Hamiltonian system (7.8) [49, 50] to the surface of the constraints (7.1) supplied with the gauge conditions $p_i - \frac{1}{2}(z\bar{z})^{-1}(z\sigma_i\mathcal{P} + \bar{\mathcal{P}}\sigma_i\bar{z}) \approx 0$.

To conclude the discussion of the twistor formulation for the charge-monopole system, let us show how the limit of zero mass described here by the Lagrangian

$$L = i\nu \frac{\bar{z}\dot{z} - \dot{\bar{z}}z}{\bar{z}z} \quad (7.15)$$

gives rise to the pure spin system. Here, as before, it is assumed that the initial configuration space is $\mathbb{C}^2 - \{0\}$. This system can be quantized by the method advocated in ref. [57], but the symplectic two-form corresponding to Lagrangian (7.15) is singular, $\omega = \omega_{ab} d\bar{z}_a \wedge dz_b$, $\omega_{ab} = 2i\nu(\bar{z}z)^{-1} \cdot (\delta_{ab} - z_a \bar{z}_b \cdot (\bar{z}z)^{-1})$, $\omega_{ab} \bar{z}_a = \omega_{ab} z_b = 0$, that complicates the analysis. There is a more short way to reveal a spin nature of the system (7.15) by showing its equivalence to the system (3.10). For the purpose we note that in the regions where either $z_1 \neq 0$ (chart U_1) or $z_2 \neq 0$ (chart U_2), one can define the complex coordinate Z as $Z = z_2/z_1$ or $Z = z_1/z_2$, respectively. Then Lagrangian (7.15) is represented equivalently as $L = i\nu(\bar{Z}\dot{Z} - \dot{\bar{Z}}Z) \cdot (1 + \bar{Z}Z)^{-1} - 2\nu\dot{\varphi}$, where φ is the phase of the coordinate z_a in the chart U_a , $a = 1, 2$. The first term coincides exactly with the Lagrangian (3.10) corresponding to the pure spin system. The second total derivative term is not important classically as well as quantum mechanically since its contribution to the action, $\Delta S = -4\nu\pi$, for periodical trajectories is trivialized, $\Delta S = 0(mod 2\pi)$, if we take into account the quantization condition $2\nu \in \mathbb{Z}$. Therefore, the system (7.15) is equivalent to the pure spin system (3.10).

The equivalence of the system (7.15) to the system (3.10) is encoded in its gauge symmetries. In correspondence with the above mentioned degeneracy of the symplectic form, the system (7.15) possesses two gauge invariances: its action is invariant under the transformations $z_a \rightarrow \rho z_a$, $\rho = \rho(t) > 0$, generated by the constraint $\rho = \bar{\mathcal{P}}\bar{z} + z\mathcal{P} \approx 0$, and $z_a \rightarrow e^{i\varphi} z_a$, $\varphi = \varphi(t) \in \mathbb{R}$, generated by the helicity constraint of the form (7.3). This means that the points z_a and ζz_a , $\zeta \in \mathbb{C}$, $\zeta \neq 0$, in configuration space are physically equivalent and should be identified, $z_a \sim \zeta z_a$. As a result, the configuration space of the system is the projective complex plane $\mathbb{C}P^1 = (\mathbb{C}^2 - \{0\})/\sim$, on which the coordinate Z introduced above plays a role of the “inhomogeneous” coordinate [27].

8 Charge-monopole in a spherical geometry and spin

Let us restrict the motion of the charge in the monopole field to the sphere $\mathbf{r}^2 = 1$. This can be done by treating the relation $\mathbf{r}^2 - 1 = 0$ as a Lagrangian constraint, $L = \frac{1}{2}\dot{\mathbf{r}}^2 + e\dot{\mathbf{r}}\mathbf{A}(\mathbf{r}) - \frac{\lambda}{2}(\mathbf{r}^2 - 1)$, whose condition of conservation generates the Hamiltonian constraint $p_r = 0$. The reduction to the surface of these second class constraints excludes the radial variables r and p_r , and we arrive at the Hamiltonian describing the charge on the sphere with the monopole in its center², $H = \frac{1}{2}(\mathbf{J} \times \mathbf{n})^2$. Here the dynamical variables \mathbf{J} and \mathbf{n} satisfy the Poisson bracket relations (2.22) and are subject to the conditions (2.23), i.e. the reduced system is a pure reduced E(3) system. The same procedure of reduction can be realized in a spinning top picture, i.e. by adding the condition $\mathbf{r}^2 - 1 = 0$ to Lagrangian (5.5) as a Lagrangian

²See also refs. [58, 59], where some aspects of the charge-monopole system in a spherical geometry were discussed.

constraint, or by putting $r = 1$ directly in Lagrangian (5.5):

$$L = \frac{1}{2}\dot{\mathbf{n}}^2 + \frac{\nu}{2}(\mathbf{e}_1\dot{\mathbf{e}}_2 - \mathbf{e}_2\dot{\mathbf{e}}_1), \quad (8.1)$$

where we suppose that $\mathbf{n} = \mathbf{e}_1 \times \mathbf{e}_2$ and vectors \mathbf{e}_1 and \mathbf{e}_2 are orthonormal. In the case of higher derivative treatment of the charge-monopole system the corresponding reduced Lagrangian is

$$L = \frac{1}{2}\dot{\mathbf{n}}^2 - \frac{\nu}{\dot{\mathbf{n}}^2}\mathbf{n} \cdot (\dot{\mathbf{n}} \times \ddot{\mathbf{n}}), \quad (8.2)$$

whereas in the twistor picture the Lagrangian takes the form

$$L = -2(\dot{z}\sigma_2 z) \cdot (\dot{\bar{z}}\sigma_2 \bar{z}) + i\nu(\bar{z}\dot{z} - \dot{\bar{z}}z), \quad \bar{z}z = 1. \quad (8.3)$$

The condition $\bar{z}z = 1$ can be omitted by normalizing appropriately the Lagrangian, i.e. by multiplying the first and second terms by $(\bar{z}z)^{-2}$ and $(\bar{z}z)^{-1}$, respectively.

The CS term is not changed by the reduction procedure due to its scale-invariance. But this term in addition is reparametrization invariant, whereas the total Lagrangian has no reparametrization invariance. We can change the non-invariant second order in velocity term $\frac{1}{2}\dot{\mathbf{n}}^2$ for the first order term $\sqrt{\dot{\mathbf{n}}^2}$, that gives rise to the reparametrization invariant action. Let us analyze the physical content of such reparametrization action considering, e.g., the modification of the Lagrangian (8.1),

$$L = \gamma\sqrt{\dot{\mathbf{n}}^2} + \frac{\nu}{2}(\mathbf{e}_1\dot{\mathbf{e}}_2 - \mathbf{e}_2\dot{\mathbf{e}}_1), \quad (8.4)$$

where $\gamma > 0$ is a dimensionless parameter. The canonical Hamiltonian of the system (8.4) is equal to zero, and from the Hamiltonian point of view, the difference of the system (8.4) from the system (8.1) consists in the presence of the constraint $\mathbf{J}^2 - \kappa^2 \approx 0$, $\kappa^2 \equiv \gamma^2 + \nu^2$, generating reparametrizations in addition to the isospin $U(1)$ gauge symmetry generated by the constraint (5.12). The system (8.4) has the same physical content as the spin system of section 3 given by only the CS term. Indeed, applying the quantization scheme of section 5, the physical subspace is separated here by Eq. (5.15) and by $(\mathbf{J}^2 - \kappa^2)\Psi_{phys}(\alpha, \beta, \gamma) = 0$. These two equations have nontrivial solutions only when the parameters κ and ν are quantized: $\kappa^2 = j(j+1)$ and $\nu = k$, where $k, |k| \leq j$, is integer (half-integer) for j integer (half-integer), and the corresponding physical state is $\Psi_{phys}(\alpha, \beta, \gamma) = \sum_{s=-j}^j C_s D_{s,k}^j(\alpha, \beta, \gamma)$, where C_s are constants. This wave function describes an arbitrary state of fixed spin j in the form alternative to the holomorphic functions of section 3. Therefore, the reparametrization-invariant system (8.4) is equivalent to the spin system described by only the CS term.

The charge-monopole system in spherical geometry can be treated as a partially gauge fixed version of the reparametrization and scale invariant spin system (8.4). To see this, we rewrite the Lagrangian (8.4) in equivalent form by introducing the einbein v :

$$L = \frac{\dot{\mathbf{n}}^2}{2v} + \frac{v}{2}\gamma^2 + \frac{\nu}{2}(\mathbf{e}_1\dot{\mathbf{e}}_2 - \mathbf{e}_2\dot{\mathbf{e}}_1). \quad (8.5)$$

Reparametrization invariance of the system (8.5) can be fixed (locally, see ref. [50]) via introducing the appropriate gauge-fixing conditions for the constraint $\mathbf{J}^2 - \kappa^2 \approx 0$, and for

the constraint $p_v \approx 0$, where p_v is the momentum canonically conjugate to v . On the other hand, introducing only the condition $v = 1$, we obtain the partially gauge fixed version of the system (8.5) [49]. The Lagrangian (8.5) with $v = 1$ is the Lagrangian (8.1) shifted for the inessential constant. Therefore, the charge-monopole system in a spherical geometry can formally be treated as a partially gauge fixed version of the spin system (8.5).

Analogously, we can change the kinetic term $\frac{1}{2}\dot{\mathbf{r}}^2$ in the initial charge-monopole Lagrangian (2.1) for the reparametrization invariant term $\sqrt{\dot{\mathbf{r}}^2}$, the latter is a kinetic term of relativistic particle in 3D Euclidean space. As a result, we get the reparametrization invariant action

$$S = \int L_r dt, \quad L_r = \frac{\dot{\mathbf{r}}^2}{2v} + \frac{v}{2} + e\mathbf{A}\dot{\mathbf{r}}. \quad (8.6)$$

Its partial gauge fixed version ($v = 1$) will give the Lagrangian coinciding up to inessential constant with the initial Lagrangian (2.1). From this point of view, the time translation, the time dilation and special conformal transformation symmetries produced canonically by the $so(2,1)$ generators H , D and \mathcal{R} [8, 38], can be treated as a relic of the reparametrization symmetry of the system (8.6) surviving the described formal Lagrangian gauge fixing procedure.

9 Charge-monopole system and anyons

Let us discuss the relationship between the charge-monopole system and (2+1)-dimensional anyons in the light of the obtained results. Earlier, the analogy with the charge-monopole system played an important role in constructing the theory of anyons as spinning particles [34]-[36].

We have observed that the charge-monopole system in many aspects is similar to the 3D free particle. In (2+1)-dimensions, spin is a (pseudo)scalar and, as a consequence, the anyon of fixed spin has the same number of degrees of freedom as a spinless free massive particle. The relationship between the charge-monopole and anyon can be understood better within the framework of the canonical description of these two systems. As we have seen, the charge-monopole system essentially is a reduced $E(3)$ system. In the case of anyons, the $E(3)$ group is changed for the Poincaré group $ISO(2,1)$. The translation generators of the corresponding groups are \mathbf{n} and p_μ , the latter being the energy-momentum vector of the anyon, and the corresponding Casimir central elements are fixed by the relations $\mathbf{n}^2 = 1$ and

$$p^2 + m^2 \approx 0, \quad (9.1)$$

where m is a mass of the anyon. The rotation (Lorentz) generators are given by \mathbf{J} and by

$$\mathcal{J}_\mu = \epsilon_{\mu\nu\lambda} x^\nu p^\lambda + J_\mu, \quad (9.2)$$

where J_μ are the translation invariant $so(2,1)$ generators satisfying the algebra of the form (2.17), $\{J_\mu, J_\nu\} = \epsilon_{\mu\nu\lambda} J^\lambda$, and subject to the relation $J_\mu J^\mu = -\alpha^2 = \text{const}$. The representation (9.2) for the anyon total angular momentum vector is, obviously, the analog of the relation $\mathbf{J} = \mathbf{L} + \mathbf{s}$ appearing under interpretation of the charge-monopole system as a particle with spin. In the charge-monopole system, the second Casimir element of $E(3)$ is

fixed either strongly, $\mathbf{Jn} = -\nu$, or in the form of the weak relation $\mathbf{Jn} + \nu \approx 0$ (see Eqs. (4.2), (4.6)). In the anyon model, spin also can be fixed either strongly, $\mathcal{J}p = Jp = -\alpha m$, or in the form of the weak (constraint) relation

$$\chi_a \equiv Jp + \alpha m \approx 0. \quad (9.3)$$

When the helicity is fixed strongly, for the charge-monopole system the symplectic form corresponding to the Poisson brackets (2.8), has a nontrivial contribution describing non-commuting quantities P_i being the components of the charge's velocity: $\omega = dP_i \wedge dr_i + \frac{\nu}{2r^3} \epsilon_{ijk} r_i dr_j \wedge dr_k$. In the anyon case, the strong spin fixing gives rise to the nontrivial Poisson structure for the particle's coordinate's [34]-[36],

$$\{x_\mu, x_\nu\} = \alpha(-p^2)^{-3/2} \cdot \epsilon_{\mu\nu\lambda} p^\lambda. \quad (9.4)$$

On the other hand, when we treat the charge-monopole system as a particle with spin (helicity is fixed weakly), one can work in terms of the canonical symplectic structure for the charge's coordinates and momenta (see Eq. (4.1)), but the canonical momenta \mathbf{p} are not observables due to their non-commutativity with the helicity constraint, whereas the gauge-invariant extension of \mathbf{p} given by Eq. (4.3) plays the role of the non-commuting quantities P_i . Exactly the same picture takes place in the case of anyon when its spin is fixed weakly: the coordinates x_μ commute in this case, $\{x_\mu, x_\nu\} = 0$, but they have nontrivial Poisson brackets with the spin constraint (9.3), whereas their gauge-invariant extension, $X_\mu = x_\mu + \frac{1}{p^2} \epsilon_{\mu\nu\lambda} p^\nu J^\lambda$ (cf. with Eq. (4.3)), $\{X_\mu, \chi_a\} \approx 0$, are non-commuting and reproduce the Poisson bracket relation (9.4). Like in the anyon case, the advantage of the extended formulation for the charge-monopole system (when we treat it as a particle with spin), is in the existence of canonical charge's coordinates r_i and momenta p_i . Within the initial minimal formulation (given in terms of r_i and gauge-invariant variables P_i), the canonical momenta p_i are reconstructed from P_i only locally, $p_i = P_i + eA_i$, due to the global Dirac string singularities hidden in the monopole vector potential. Having in mind the gauge invariant nature and non-commutativity of P_i or their analogs Π_i from the extended formulation, we conclude that they, like anyon coordinates X_μ , are the charge-monopole's analogs of the Foldy-Wouthuysen coordinates of the Dirac particle [36, 60].

10 Concluding remarks

The discussed classical $so(2,1)$ symmetry can be quantized in an abstract way proceeding from the classical relations (2.17) and (2.19). In such a way the infinite-dimensional unitary half-bounded $sl(2, R)$ representations of the discrete series D_α^+ will be obtained, which are characterized by the quantum Casimir element $\mathcal{C} = -\alpha(\alpha - 1)$, $0 < \alpha \in \mathbb{R}$ and by the eigenvalues $j_0 = \alpha + n$, $n = 0, 1, \dots$, of the operator \mathcal{J}_0 [43]. Since $\alpha > 0$ is arbitrary, such a quantization procedure of the $so(2,1)$ symmetry algebra does not introduce any restrictions for the charge-monopole coupling constant, and does not fix correctly the spectrum of the operator \mathbf{J}^2 . The latter information, as we saw, is encoded in the corresponding $so(3)$ algebra and classical condition $\mathbf{J}^2 > \nu^2$. The quantization of the parameter ν and the quantum spectrum of \mathbf{J}^2 could be obtained in principle by applying the geometric quantization to the

classical $E(3)$ system from section 2.4. On the other hand, the same information could be extracted from the quantization of classical $so(3, 1)$ symmetry of the charge-monopole system described in section 2.3 with taking into account the classical relation $\mathbf{J}^2 > \nu^2$. We are going to consider the geometric quantization of $so(3, 1)$ charge-monopole symmetry elsewhere. The observation of the similarity between the charge-monopole and the 3D free particle systems realized in section 2.3 in the context of the vector integrals of motion has been applied recently in ref. [61] for explaining the nature of the nonstandard fermion-monopole supersymmetry [62].

In sections 4 and 5 we have discussed the two different interpretations of the charge-monopole system as a particle with spin and as a spinning top system. It is interesting to find the corresponding map between these pictures at the Hamiltonian level. In the first picture, the “rotational” part of the phase space is given by the spin vector \mathbf{s} , $\mathbf{s}^2 = \nu^2$, and by the orbital angular momentum \mathbf{L} and associated unitary vector \mathbf{n} . In the spinning top picture we have the angular momentum vector \mathbf{J} and the set of orthonormal vectors \mathbf{e}_a . The vector \mathbf{J} of the second formulation is identified with the total angular momentum vector $\mathbf{L} + \mathbf{s}$ from the first formulation, and the vector \mathbf{e}_3 giving the symmetry axis of the top is naturally identified with \mathbf{n} . Therefore, to find the mapping between the two formulations, it is sufficient to construct from the variables \mathbf{s} , \mathbf{L} and \mathbf{n} the orthonormal vectors $\mathbf{e}_{1,2}$, $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{n}$, satisfying the necessary Poisson bracket relations. Unfortunately, we did not succeed in realization of such a construction.

It seems interesting to investigate analogously other systems of particles (strings) in the background of external gauge or gravitational fields from the point of view of their possible alternative description as the particles (strings) with internal reduced degrees of freedom.

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